Order-sorted logic meets intensional logic

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KU Leuven, DTAI seminar

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Context: Knowledge Representation and Logic

Knowledge Representation and Logic

Motivation:

- Formal representation of knowledge.

"All man are mortal." $\forall x : Man(x) \Rightarrow Mortal(x).$

"Socrates is a man." $Man(Socrates).$

- Formal reasoning methods.

"Therefore, Socrates is mortal." $Mortal(Socrates).$

- Exactness, Explainability

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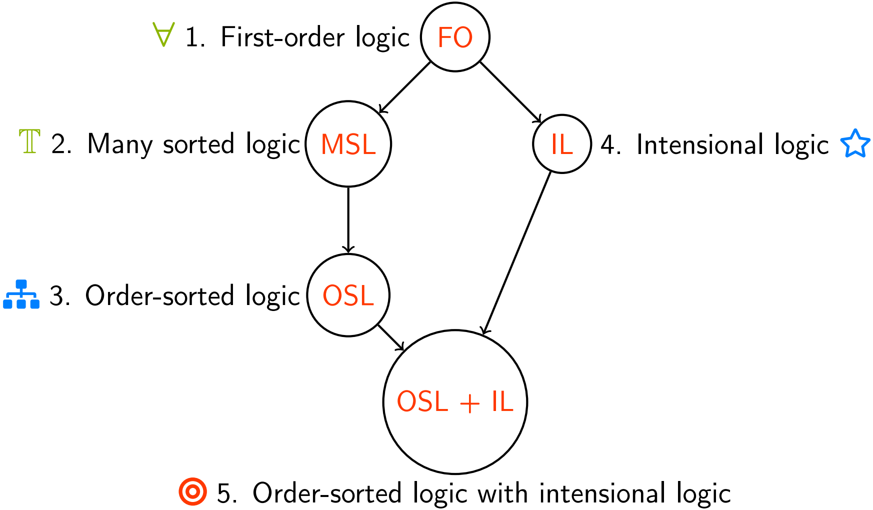
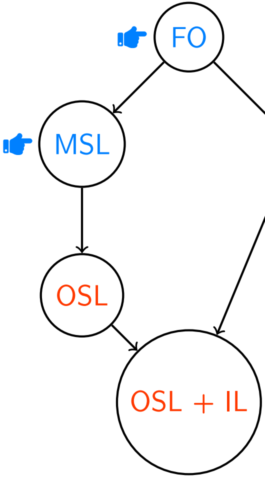
Roadmap


1. First-order logic
FO
2. Many sorted logic
MSL
IL
4. Intensional logic
3. Order-sorted logic
OSL
OSL + IL
5. Order-sorted logic with intensional logic

$\forall x \in \mathbb{T}$ Forst-order and many sorted logic



Knowledge Representation and Logic

(Un)-sorted first-order logic

- Example – Model dynamic system of animals:

  "At any *Time* point there is a *Cat* meowing."

- First-order logic is not typed! Each object is part of "universe".

$$\forall t : Time(t) \Rightarrow (\exists c : Cat(c) \wedge Meow(t, c)).$$

- In first-order **typed** logic (many-sorted logic), we can introduce types *Time* and *Cat*.

$$Meow : (Time, Cat) \rightarrow \mathbb{B}$$

$$\forall t[Time] : \exists c[Cat] : Meow(t, c).$$

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Knowledge Representation and Logic

Sorted first-order logic

- Suppose there are types: *Time*, *Cats*, and *Dogs*.
- Let's model the age of animals, how?

$$CatAge : (Time, Cat) \rightarrow \mathbb{N}$$

$$DogAge : (Time, Dog) \rightarrow \mathbb{N}$$

- At the current moment (*now*), all animals have age less then 50.

$$\forall d[Dog] : DogAge(now, d) < 50.$$

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Knowledge Representation and Logic

Sorted first-order logic

- How we would like it to be?

$$Age : (Time, Cat \cup Dog) \to \mathbb{N}.$$

- At the current moment (*now*), all animals have age less then 50.

$$\forall a[Cat \cup Dog] : Age(now, a) < 50.$$

- It would be nice if there was a type *Animal* containing both *Cat* and *Dog*.

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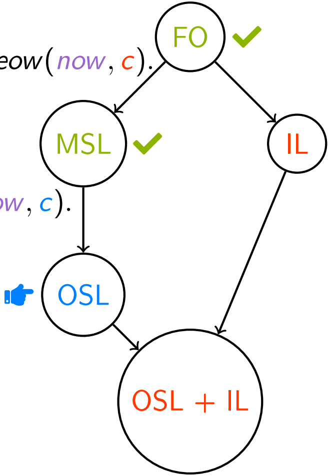
Order-sorted logic

$\exists c : Cat(c) \wedge Meow(now, c).$

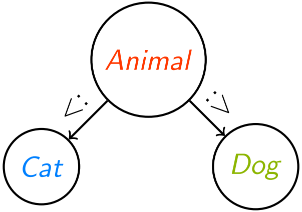
$\exists c[Cat] : Meow(now, c).$



Order-sorted logic

- Hierarchy of types ($<:$ subtype relation):

$$Dog <: Animal \quad Cat <: Animal$$



- Liskov substitution principle!

$$S <: T \Rightarrow (\forall x[T] : \phi(x)) \Rightarrow (\forall y[S] : \phi(y))$$

- Assumptions:
  - Non-emptiness:

$$\forall T : \exists x : T(x)$$

  - Disjoint types:

$$Dog \cap Cat = \emptyset$$

  - Non partitioning:

$$Dog \cup Cat \subseteq Animal$$

  - Predicative form:

$$Dog : (Animal) \rightarrow \mathbb{B}$$

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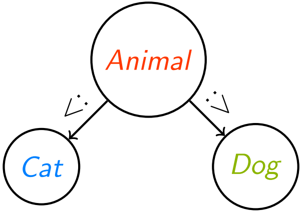

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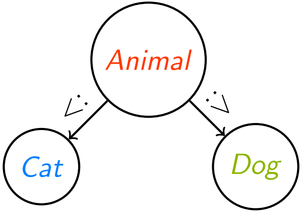

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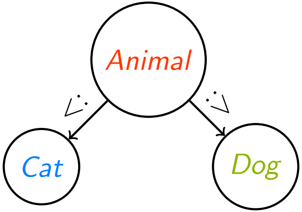

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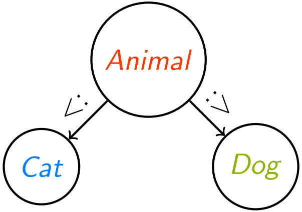

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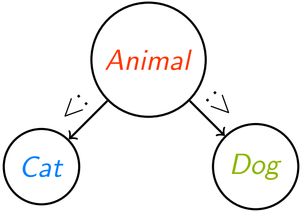

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- Back to the age example; how we would (really) like it to be?

$$Dog <: Animal$$

$$Cat <: Animal$$

$$Age : (Time, Animal) \to \mathbb{N}$$

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- There is a Cat at this moment (*now*) of age 22.

$$\exists c[Cat] : Age(now, c) = 22.$$

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- The same setup:

$$Dog <: Animal$$

$$Cat <: Animal$$

- Predicative form:

$$Dog : (Animal) \to \mathbb{B} \qquad Cat : (Animal) \to \mathbb{B}$$

- All animals are either dogs or cats!

$$\forall a[Animal] : Dog(a) \lor Cat(a).$$

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- Let's get back to the sounds of animals.

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$$Bark : (Time, Dog) \rightarrow \mathbb{B} \qquad Meow : (Time, Cat) \rightarrow \mathbb{B}$$

- At any time point there is a cat meowing.

$$\forall t[Time] : \exists c[Cat] : Meow(t, c).$$

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- The new vocabulary:

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$$ProducingSound : (Time, Animal) \to \mathbb{B}$$

- How to define *ProducingSound*?

$$\forall t[Time] : \forall a[Animal] : ProducingSound(t, a) \Leftrightarrow$$
$$(\exists d[Dog] : a =^{*1} d \land Bark(t, d)) \lor$$
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- How to define *ProducingSound*? Second attempt!

$$\forall t[Time] : \forall a[Animal] : ProducingSound(t, a) \Leftrightarrow$$
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Guarded Order-sorted logic

- Consider types S and T such that:

$$S <: T$$

$$S : (T) \to \mathbb{B}$$

- Given information that term a is of type T as $(a : T)$:

$$S(a) \Rightarrow \phi(a) \qquad \text{and} \qquad S(a) \wedge \phi(a)$$

  Are well typed formulas!
- We abbreviate these with:

$$\langle\langle \phi(a) \rangle\rangle \qquad \text{and} \qquad [[\phi(a)]]$$

- Which terms to guard first?

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Order-sorted logic

- Let's get back to the sounds of animals.
- At the current time point (*now*), there is an animal producing sound.

$$\exists a[Animal] : ProducingSound(now, a).$$

- With *ProducingSound* being defined as:

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- We defined *ProducingSound* as:

$$\forall t[Time] : \forall a[Animal] : \underline{ProducingSound}(t, a) \Leftrightarrow (Dog(a) \land Bark(t, a)) \lor (Cat(a) \land Meow(t, a)).$$

- How *ProducingSound* could be defined better? In natural language:

  At any time point, an animal produces sound (at that time point) iff it produces a sound characteristic for its species.

- But what does this mean? For example:

  If it is a Dog it Barks, if it is a Cat it Meows, ...

- Hence, sound characteristic for species is collection of **concepts** as Barks, Meows, ...

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☆ Intensional logic

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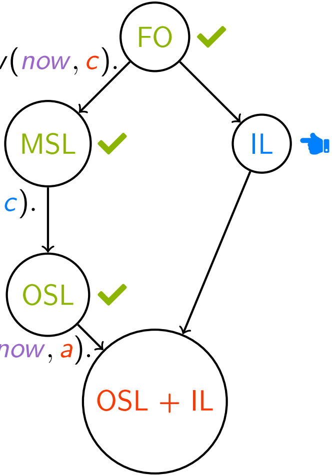
$\exists c[Cat] : Meow(now, c).$

$\exists a[Animal] : ProducingSound(now, a).$



Intensional logic

- Concepts are not the same as objects!
- Evening star and morning star are concepts.
  - Celestial bodies they are denoting are objects.
  - In turned out that both are (planet) Venus!
- Sometimes distinction is important!
  - In this case there are **two** concepts but **one** object.

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  - In turned out that both are (planet) Venus!
- Sometimes distinction is important!
  - In this case there are **two** concepts but **one** object.

Intensional logic

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- Sometimes distinction is important!
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Intensional logic

Example vocabulary:

$$CelObject = \{Sun, Moon, \ldots, Venus, \ldots\}$$
$$morningStar : CelObject$$
$$eveningStar : CelObject$$
$$observableCelObject : (Concept) \to \mathbb{B}$$

State of affairs:

$$morningStar = Venus.$$
$$eveningStar = Venus.$$
$$observableCelObject = \{`morningStar, `eveningStar\}$$

Example theory:

$$\#\{c[Concepts] \mid observableCelObject(c)\} \geq \#\{co[CelObject] \mid \exists c[Concepts] : observableCelObject(c) \wedge \$(c)() = co\}.$$

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Intensional logic

- Back to the sounds of animals!
- How *ProducingSound* could be defined better? In natural language:

  At any time point, an animal produces sound (at that time point) iff it produces a sound characteristic for its species.

- Hence, sound characteristic for species is collection of **concepts** as Barks, Meows, ...
- The new vocabulary:

$$Dog <: Animal \qquad Cat <: Animal$$

$$Dog : (Animal) \to \mathbb{B} \qquad Cat : (Animal) \to \mathbb{B}$$

$$Sound(Concept) \to \mathbb{B}$$

$$Bark : (Time, Dog) \to \mathbb{B} \text{ isa } Sound$$

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$$\forall t[Time] : \forall a[Animal] : \underline{ProducingSound}(t, a) \Leftrightarrow \exists s[Concept] : Sound(s) \wedge \$(s)(t, a).$$

- Is this good? No!

$$\forall t[Time] : \forall a[Animal] : \underline{ProducingSound}(t, a) \Leftrightarrow Bark(t, a) \vee Meow(t, a).$$

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Intensional logic

- Can this be made correct in intensional logic?

$$\forall t[Time] : \forall a[Animal] : \underline{ProducingSound}(t, a) \Leftrightarrow Bark(t, a) \lor Meow(t, a).$$

- Yes!
- Some tips:

$$\begin{aligned} Sound &: (Concept) \to \mathbb{B} && := \{Bark, Meow\} \\ Species &: (Concept) \to \mathbb{B} && := \{Dog, Cat\} \\ SoundToSpecies &: (Concept) \to Concept && := \{(Bark \to Dog), (Meow \to Cat)\} \end{aligned}$$

- Something like:

$$\begin{aligned} \forall t[Time] : \forall a[Animal] : &\underline{ProducingSound}(t, a) \Leftrightarrow \\ &\exists so[Concept] : \exists sp[Concept] : Sound(so) \land Species(sp) \land \\ &SoundToSpecies(so) = sp \land \$(sp)(a) \land \$(so)(t, a). \end{aligned}$$

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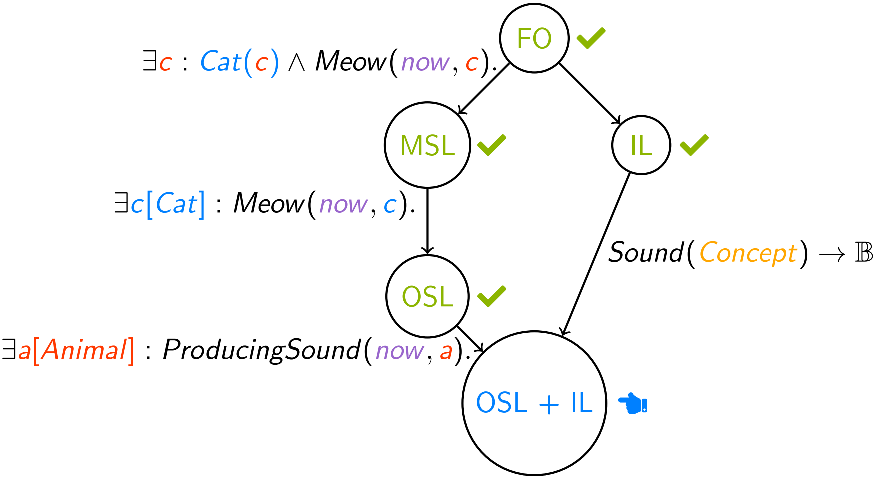
Order-sorted logic with intensional objects



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- Which was abbreviated as:

$$\forall t[Time] : \forall a[Animal] : \underline{ProducingSound}(t, a) \Leftrightarrow [[Bark(t, a)]] \lor [[Meow(t, a)]]$$

- Introduce new type:

$$Sound <: Concept \quad := \{Bark, Meow\}$$

- Then *ProducingSound* could be defined as:

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Order-sorted logic with intensional objects

- *ProducingSound* defined in natural language:

  At any time point, an animal produces sound (at that time point) iff it produces a sound characteristic for its species.

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Order-sorted logic with intensional objects

- Another example:

  At the current time point (*now*), fore each species there is an animal producing sound appropriate for that species.

- Expressed in OSL+IL as:

$$\forall s[Sound] : \exists a[Animal] : [\![\$(s)(now, a)]\!].$$

- Which reduces to:

$$(\exists a[Animal] : Dog(a) \land Barks(now, a)) \land (\exists a[Animal] : Cat(a) \land Meow(now, a)).$$

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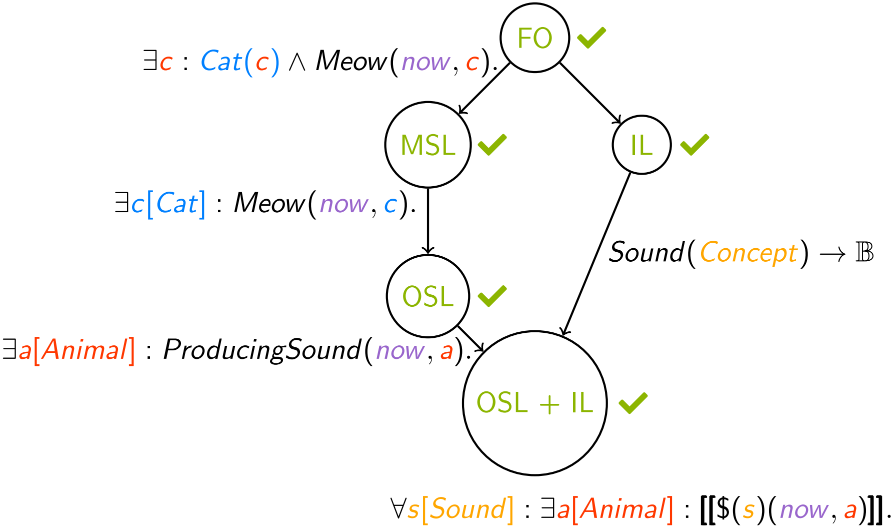
Conclusion


FO
∃c : Cat(c) ∧ Meow(now, c).
MSL
IL
∃c[Cat] : Meow(now, c).
Sound(Concept) → 𝔹
OSL
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OSL + IL
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Conclusion

- First-order logic and many sorted logic are sometimes restrictive.

  At the current moment (*now*), all animals have age less then 50.

- Order-sorted logic resolves some issues but also introduces others.

  At the current time point (*now*), there is an animal producing sound.

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  At any time point, an animal produces sound (at that time point) iff it produces a sound characteristic for its species.

- OSL combined with Intensional logic seems to resolve some of these issues nicely.

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Thank you for your attention!

Questions?

Djordje Markovic [*dorde.markovic@kuleuven.be*]

28. November 2023